Version 1.0

Introduction to the Geometry for Teachers course Student Learning Outcomes

What should preservice teachers learn in a college geometry course, and how should they learn it? There has been little nationwide consensus on how best to answer this two-fold question, despite the importance of high school geometry in college preparatory curricula. The geometry for teachers (GeT hereafter) courses taught at colleges and universities across the United States demonstrate a wide variety of content focus, reflecting institutional demands, curricular requirements, and instructor interest (Grover & Connor, 2000; Venema et al., 2015).

In 2017, the University of Michigan GRIP[1] lab, under the direction of Dr. Patricio Herbst and Dr. Amanda Brown, received a National Science Foundation grant for the project "GeT Support: An online professional learning community to support the geometry course for teachers." The grant grew out of Herbst and Brown's commitment to improving preservice teacher (PST) preparation, especially in the content area of geometry. They recruited GeT course faculty from across the country with the intent of creating a community of instructors sharing that commitment. The *Get: a Pencil* Community first convened in Ann Arbor in June 2018, and has continued to have a robust online presence since then. The ongoing focus of *GeT: a Pencil* is to work collectively towards the goal of improving the capacity of secondary geometry teaching.

Since the conference in 2018, the community has continued to meet online to discuss issues relevant to the GeT course. One issue that surfaced early in these discussions was the lack of a clear shared understanding of what should be in the GeT course. A subgroup of the community, the Teaching GeT working group, was formed to try to address this issue, and it is this subgroup that has been responsible for putting together this document. Through our initial conversations, we (the working group) realized that in order to better prepare preservice teachers to teach geometry, we needed a common set of content objectives for the GeT course. Everyone in the Teaching GeT working group had an opportunity to contribute their own list of what content they felt should be included in a GeT course. We spent months meeting online, discussing all suggestions, respectfully disagreeing on items, and narrowing the list down to ten essential student learning objectives (SLOs). Here, *essential* means the identification of content knowledge that all prospective secondary geometry teachers should have the opportunity to learn. For each of these SLOs, there is a brief content description, a paragraph with more details, and a longer narrative that describes the SLO in detail, along with suggestions of specific content to include when covering the SLO.

1. Proofs
2. Critique Reasoning
3. Secondary Geometry Understanding
4. Axiomatic Systems
5. Definitions
6. Technologies
7. Euclid
8. Constructions

[1] Grasping the Rationality of Instructional Practices

9. Non-Euclidean Geometries
10. Transformations

One of the tensions that we faced during the creation of the SLOs was whether or not we should prescribe specific teaching practices for a GeT course. On the one hand, we want to emphasize recommendations for college mathematics instructional practices, such as those espoused by the Mathematical Association of America (Abell et al., 2017), that are particularly relevant to geometry teaching. GeT courses are great places for active and inquiry-based styles of learning, and there are many accessible kinds of problems to be solved. On the other hand, we want to highlight opportunities for GeT instructors to facilitate students' learning of secondary teaching practices, such as those recommended by the NCTM (2020) [See SLO 3]. Because the audience of the GeT course and the course itself currently varies so much between institutions, we want these SLOs to be flexible. One of these flexibilities is that they may be used in a single course or spread throughout various courses in a program. Therefore, we support a pedagogical frame whereby faculty using the SLOs have the academic freedom to make informed decisions regarding the teaching methods used in their programs and courses. Thus, the narratives of the SLOs include references to pedagogical resources and content to make faculty teaching the GeT course aware of these and where to find them.

The group has identified some best practices for the GeT course that we believe should apply to all courses. Some of these practices are specifically outlined in the SLOs. For instance, geometry is a traditional setting for teaching proof, but it is more generally an ideal setting for working on all types of mathematical communication. [See SLO 1 on Proof and SLO 2 on Critiquing Reasoning.] It is important for students to work together to solve problems, and at the same time learn to productively collaborate. Geometry courses are also an ideal setting for allowing students to experience the progression of exploring followed by conjecturing followed by proving. This central mathematical process is one that students may not experience in very many classes in their college career. Geometry courses should introduce Dynamic Geometry Environments [See SLO 6] because they afford students vital opportunities to explore and develop conjectures which can be proven or disproven.

Other times our recommended practices may not link directly to specific SLOs. They instead regard general processes for learning mathematics that we believe students should experience within a GeT course in order to better understand the geometry content. Applying geometry to contexts outside of mathematics and connecting geometry to other mathematical domains are each valuable. Students should have many chances to experience and develop proficiency with the mathematical process skills of problem-solving, oral and written communication of mathematical ideas, and productive collaboration. Some curricula focus on the exploration and conjecturing, while others focus on the proving, but we think that it is most powerful for students to go through this entire process and to try to prove their own conjectures or produce counterexamples. This requires planning and knowledge on the part of the instructor but is well worth it.

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SLO 1: Derive and explain geometric arguments and proofs.

Summary

Proof is a cornerstone of mathematics, and a GeT course should enhance a student's ability to read and write proofs of theorems, apply them, and explain them to others. Geometry offers one of the best opportunities for students to take ownership of proof in both the undergraduate and secondary curriculum. Because geometric proofs come with natural visualizations, they provide a rich environment for students to think deeply about the arguments that are being made and to make sense of each statement. Students should understand that proof is the means by which we demonstrate deductively whether a statement is true or false, and that geometric arguments may take many forms. They should understand that in some cases one type of proof may provide a more accessible or understandable argument than another.

Narrative

Proof is at the heart of mathematics, and geometry is a place that offers robust opportunities for students to take ownership of proving in both the undergraduate and secondary curriculum. Geometric proofs are the result of investigation and conjecture, often stemming from natural visualizations; therefore, they offer the opportunity to think deeply about the arguments that are being made and try to understand what is being communicated. Prospective geometry teachers need to be able to understand different types of proofs, such as synthetic (from axioms), analytic (using coordinates), and proofs using transformations or symmetries. They should also be able to communicate proofs in different ways (two-column, paragraph, or a sequence of transformations).

Proof is a central idea in many state K-12 mathematics standards. While not every state has adopted or uses the Common Core State Standards for Mathematics (CCSSM), many have written new mathematics standards that are very similar to these standards. In the CCSSM, the first mention of proof is in the eighth grade geometry standards. These standards focus on using rotations, translations, and reflections to demonstrate that two figures are congruent to one another, and students are expected to be able to explain a proof of the Triangle Angle Sum, as well as the Pythagorean Theorem and its converse. In the CCSSM high school geometry standards, students are expected to "understand congruence in terms of rigid motion" and "prove geometric theorems" (National Governors Association for Best Practices[NGA] & Council of Chief State School Officers [CCSSO], 2010).

For secondary mathematics teachers to be prepared to teach proof, they need to have numerous and varied experiences to develop a deep understanding of proof. Mathematics teachers need to experience geometry proofs from the student's perspective so they can empathize when their own students struggle. It is also essential that they can choose the most accessible type of proof for the situation.

The geometer David Henderson (2006) argued that we should be teaching “alive mathematical reasoning,” in which we view proofs as “convincing communications that answer—Why?” (p. 13). Under this view, the role of a proof is not merely to show that something is true but to clearly communicate to others *why* it is true. Most mathematicians would agree that explaining why something is true is an important value in mathematics, but it is not a perspective that we always share with our students when teaching them about proof. There are sometimes tradeoffs involved in choosing how to present material between making it clear why something is true and using easily adaptable proof methods. It is valuable to show GeT students a range of possibilities and ask them to evaluate the pros and cons as a class.

The reasoning used in arguments can be deductive or inductive. An argument that uses deductive reasoning, a deductive argument, is one that proceeds from a given set of assumptions to the logical consequences of those assumptions, while an argument that uses inductive reasoning, an inductive argument, is one where a pattern is noticed and assumed to continue. In mathematics, we primarily value deductive arguments, and proofs must be deductive. However, in most other disciplines arguments are largely inductive, so many students may be at first unaware that we are looking for deductive proofs. Students need to understand the difference between observations that they can make from dynamic geometry software [See SLO 6] and deductive arguments that show that something is always true.

Another distinction is often made between synthetic proofs and analytic proofs. Synthetic proofs are those that are made in the style of Euclid, in which we make arguments from axioms without coordinatizing the points, lines, and other geometric objects being studied. If we coordinatize these geometric objects by writing down equations for them, we arrive at analytic geometry. Analytic geometry is covered extensively in algebra and calculus classes, and so most geometry classes focus on synthetic geometry.

It is also important for students to understand in order to prove something is not true, they need to find a counterexample. It can be hard for them to realize that if a proof shows **for all** cases where the hypothesis is true then the conclusion is also true. However, to show a conditional statement is not true, they just have to find one case where the hypothesis is true but the conclusion is false. Finding counterexamples is important to their development of a robust understanding of proof.

In a GeT course, proof can take several forms. It could be very informal or very formal using a given set of axioms. It could be written in paragraph form or as a two-column proof. Students often enter a GeT course feeling scared of the word “proof” and need support to learn that it just means a convincing argument that something is true. Writing a synthetic proof requires the ability to construct a logical argument in a systematic manner. This skill leads to growth in critical thinking and reasoning. Daily, we encounter situations where critical thinking is needed.

Geometry is a great course to use proof to help students build problem-solving skills, which is why proof should be at the heart of the GeT course.

References

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SLO 2: Evaluate geometric arguments and approaches to solving problems.

Summary

Students should have opportunities not only to write proofs, but also to evaluate proofs and other types of reasoning. Geometry courses provide a natural setting for students to reflect on their own reasoning, share their reasoning with one another, and critique the reasoning of their peers. If students only ever see correct arguments given by teachers and textbooks, they may not carefully evaluate the arguments because they assume they will always be correct. The ability to evaluate other people's arguments is an important real-world skill that is related to, but separate from, the skill of proof writing. Critiquing reasoning is a competency that needs to be practiced in order to improve it and is an essential skill for future geometry teachers.

Narrative

Geometry courses provide a natural setting for students to reflect on their reasoning, share their reasoning, and critique the reasoning of their peers. If students only ever see correct arguments given by teachers and textbooks, they may not learn how to critically evaluate arguments. The ability to evaluate other people's arguments is an important real-world skill that is related to, but separate from, the skill of proof writing.

Critiquing reasoning is a competency that needs to be practiced in order to improve and is an essential skill for future geometry teachers. Students should have opportunities to critique reasoning throughout a geometry course. This can take on many forms: critiquing their own or other students' proofs, working together in groups to solve a problem, classroom discussions of problems or proofs, posing problems that lead to student disagreement, and learning about the geometric thought process in the Van Hiele Levels.

Providing opportunities for students to critique geometric reasoning is also important for understanding nuances in geometric definitions [see SLO 5] and in geometric notation. An essential opportunity for students to practice critiquing reasoning is when GeT students present their reasoning and proofs to the class, with the instructor modeling and moderating positive and negative feedback and depth of questioning. Some instructors have also found it valuable to introduce others' arguments that could arise in high school geometry contexts, such as sample student proofs or video approximations of secondary teaching situations. Instructors can position GeT students in the role of a teacher, rather than a peer, and invite broad and diverse observations about "students'" thinking. These discussions allow GeT students to practice critiquing reasoning while simultaneously deepening their understanding of secondary geometry content [see SLO 3].

While some of this reasoning will be in the form of formal proofs, other reasoning will be much more informal. Class discussions and work solving problems in groups can be great opportunities for students to practice critiquing the work of others. If groups are solving non-routine problems together, there are bound to be many opportunities for students to discuss their reasoning and to listen to and evaluate the reasoning of their peers. Likewise, whole-class discussions are further opportunities for students to hear and evaluate the reasoning of their peers. In an inquiry-based geometry classroom, one of the goals of the instructor is likely to be to create an environment in which students feel supported in sharing their reasoning and their thoughts about other people's reasoning in a constructive way.

If a classroom environment is achieved in which people feel safe to share their reasoning and their thoughts about other people's positions, then genuine disagreements may arise in the class. When this happens, it can be one of the situations that is most conducive to deep learning. When students feel that they have an emotional stake in the outcome of a discussion, they start paying deep attention to arguments on both sides. Instructors can look for problems to pose that are likely to lead good students to disagreements. Non-Euclidean geometries [see SLO 9] are particularly fertile for leading to productive disagreements, as everything there will be new and unfamiliar, and it will take some time to reach an agreement as to how to proceed. For example, students can try to decide which of Euclid's Postulates and/or Propositions [see SLO 7] are true in a new geometry or what familiar definitions [see SLO 6] give rise to in other geometries. Any situation where students are making conjectures and trying to evaluate if they are true will lead to opportunities for students to come up with competing ideas they will need to resolve.

Interestingly, GeT instructors anecdotally have reported that a growing number of college students are exhibiting gaps in their geometric understandings and that students in their college classrooms sometimes struggle with visualizing relationships among quadrilaterals and have difficulties characterizing them. The Van Hiele levels are levels of learners' geometric thinking and understanding (Mason, 1998). The five sequential levels include Visualization, Analysis, Abstraction, Deduction, and Rigor. This model of geometric learning posits that students at all levels will move through these different levels each time they encounter a new geometric subject. Although we expect preservice teachers to reach a high geometric thinking level (level 4 or 5), students who enter a high school geometry class typically perform at the lower levels. Therefore, it is recommended that GeT instructors create opportunities for preservice teachers to critique reasoning at various thinking levels. While it is natural for group activities to provide opportunities for the analysis of reasoning, the use of individual assignments can also be useful. For example, the use of end-of-class self-reflection assignments can provide GeT instructors feedback regarding gaps in student understanding or provide evidence of creative thinking and insightful connections.

Some other ways that instructors have implemented this standard in their classrooms include:

- Grading mock proofs on a test;
- Having students create rubrics for an assignment and evaluate their own work;
- Looking at possible K-12 classroom activities and asking students to critique them and discuss them in reference to their own future teaching; and

- Doing a jigsaw, think/pair/share, or “speed dating” activities discussing proof.

Regardless of the form that critiquing takes on, it is an essential aspect of a GeT course as it helps students think critically, improve their reasoning skills, learn how to develop solid mathematical arguments, and become better mathematicians.

References

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<u>SLO 3: Understand the ideas underlying current secondary geometry content standards and use them to inform their own teaching.</u>

Summary
Future secondary geometry teachers must deeply understand specialized content that is aligned to national and state secondary standards, know the best practices for teaching the content, and be able to reflect on their teaching. Due to limited time and instructors' varied preferences in content selection, it is not practical to suggest a list of geometry topics to be covered in a GeT course. Thus, the GeT course should focus on helping students understand essential ideas emphasized in secondary geometry standards and use them to inform their future teaching. GeT instructors should be able to incorporate teacher preparation standards into their course designs in a way that fits their teaching agenda and introduce the national and state curriculum standards to future teachers. In addition, a GeT course should foster the construction of pedagogical content knowledge by sharing teaching techniques and by engaging students in conversations about teaching geometry content.

Narrative
While the Geometry for Teachers (GeT) course at most institutions contains students who do not plan to teach, it is required for those who will become secondary math teachers. To be a good secondary geometry teacher, one must understand the content, know the best practices for teaching the content, and be able to reflect on one's teaching.

Although high school geometry is described as "devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates)" (NGA & CCSSO, 2010, para. 2), for many reasons, students in the U.S. often enter a GeT course with varying levels of knowledge in Euclidean geometry. As GeT instructors, it is our job to fill in the students' knowledge gaps so they are prepared to teach secondary geometry. However, due to limited time in a GeT course (usually one semester) and GeT instructors' varied preferences in content selection, it is not practical to suggest a list of geometry topics to be covered in a GeT course. Thus, the GeT course should focus on helping students understand essential mathematical practices and develop problem-solving skills that can be applied to the variety of geometry topics they may find themselves teaching.

The Standards for Mathematical Practices are an important piece of the Common Core State Standards for Mathematics (CCSSM), which "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (NGA & CCSSO, 2010, para 1). Even though some states have moved away from using CCSSM and have developed their own state standards, their new state standards typically include these eight practices or something similar to them. These practices form the foundation for good mathematics teaching. They are

1. *Make sense of problems and persevere in solving them.*
2. *Reason abstractly and quantitatively.*
3. *Construct viable arguments and critique the reasoning of others.*

4. *Model with mathematics.*
5. *Use appropriate tools strategically.*
6. *Attend to precision.*
7. *Look for and make use of structure.*
8. *Look for and express regularity in repeated reasoning.* (NGA & CCSSO, 2010)

Furthermore, these practices provide the structure for mathematical problem solving, and any GeT student can benefit from becoming a better problem solver. We also want pre-service secondary geometry teachers to be able to model these practices in their future classrooms so GeT instructors should model these in our own classrooms.

All GeT instructors need to be aware of teacher preparation standards that have been created to help prepare secondary geometry teachers (Table 1) and incorporate them into their GeT course designs in a way that fits their teaching agenda. Many different professional organizations (e.g., AMTE and NCTM) have contributed to these standards and suggested what faculty should be doing to prepare better secondary mathematics teachers. GeT instructors should also be aware of their state and national standards (Table 1) and introduce them to GeT students so that they can start to become familiar with the standards that they will teach. Many states in the U.S. have adopted the CCSSM for their K-12 schools (see this map), and if your state does not use CCSSM, it is best to Google "State K-12 Mathematics Standards."

Table 1

Resources for Standards

Standards		**Issuing Organizations**
Teacher Preparation Standards	*The Mathematical Education of Teachers II* (2012)	Conference Board of the Mathematical Sciences
	Standards for Preparing Teachers of Mathematics (2017)	Association of Mathematics Teacher Educators
	Standards for the Preparation of Secondary Mathematics Teachers (2020) *Standards for the Preparation of Middle-Level Mathematics Teachers* (2020)	National Council of Teachers of Mathematics

Curriculum Standards for K-12 Schools	*Principals and Standards for School Mathematics* (2000)	National Council of Teachers of Mathematics
	Common Core State Standards for Mathematics (CCSSM) (2010)	National Governors Association Center for Best Practices, Council of Chief State School Officers

Because GeT courses are taken by preservice mathematics teachers, GeT instructors must understand the needs of this group of students in their course. It is not enough for preservice teachers to know the content; these students must also gain specialized pedagogical knowledge to teach effectively. Shulman (1986) describes this as pedagogical content knowledge; it includes, in part, an understanding of what makes learning some topics easy or difficult. To have this type of understanding, students must have opportunities to reflect upon and compare/contrast analogies, illustrations, and examples. Ball, Thames, and Phelps (2008) describe pedagogical content knowledge as a bridge between content knowledge and the practice of teaching. GeT instructors should foster the construction of this knowledge by sharing teaching techniques and through conversations about teaching geometry content. For example, by taking the time to discuss multiple approaches to solving problems or by examining different frameworks for writing proofs, the GeT instructor is providing students the opportunity to reflect on misconceptions and ways that make the content more understandable to others. This type of knowledge is necessary for future teachers.

References

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<u>SLO 4: Understand the relationships between axioms, theorems, and different geometric models in which they hold.</u>

Summary
Geometry courses are one of the few places where students have opportunities to engage explicitly with axiomatic systems. Mathematical theories can be developed from a small set of axioms with theorems proven from those axioms. However, GeT students may have limited prior experience working with axiomatic systems. Students in GeT courses should gradually develop the ability to:

(a) recognize and communicate the distinction between axioms, definitions, and theorems, and describe how mathematical theories arise from them,
(b) construct logical arguments within the constraints of an axiomatic system, and
(c) understand the roles of geometric models, such as the plane, the sphere, the hyperbolic plane, *etc*., in identifying which theorems can or cannot be proven from a given set of assumptions.

Narrative

Most teachers of college geometry classes will be familiar with the idea that it is possible for mathematical subjects to be reduced to a set of axioms and then a set of theorems proven (exclusively) from those axioms. However, even though this can be done in principle, it is something that working mathematicians rarely do outside of geometry and logic classes, so not everyone teaching a geometry class will have a lot of experience working with axiom systems.

In building an axiom system, we begin with **undefined terms,** as well as with statements that are accepted to be true without proof called **axioms**. In geometry, the undefined terms include “points” and “lines.” The axioms establish assumptions about undefined terms and the relationships between them [see SLO 5 on definitions]. A world in which we can give meanings to all of those terms is called an **interpretation** of the terms[2]. For example, if we are talking about “points” and “lines,” we could interpret them as points and lines in the Euclidean plane, but we could also interpret them as points and great circles on the sphere. An interpretation is called a **model** of the axioms if all of the axioms are true in the interpretation. For example, if we have an axiom that says that two points lie on a unique line, the Euclidean plane would be a model of this axiom. The sphere would not, because antipodal points like the north and south poles on the sphere can be connected by many different great circles [see SLO 9 on non-Euclidean Geometry].

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[2] The definitions of terms given in this section are standard in mathematical logic. See, for example (Kleene, 1971; Ebbinghaus, Flum, and Thomas, 1994). The branch of logic that deals with models and interpretations is called “Model Theory,” which is not to be confused with “Mathematical Modeling,” which is a completely different subject in which the word “model” is used to mean something else. See (Hodges, 2022) for more on this.

Axioms within a system are **independent** if no axiom in the system is a logical consequence of the others. This means that for any axiom, we should be able to find an interpretation in which that axiom fails, but all of the others are true. Models and independence are intimately tied into the history of geometry [see SLO 7]. Perhaps the biggest question about Euclid's axiom system was whether his fifth postulate could be proven from his first four axioms. Although an apparently consistent hyperbolic geometry was developed in the early 1800s, it was not until Beltrami presented a model for that system later in the century that the independence of the parallel postulate was established.

A **theorem** is a statement that has been proven from the axioms without regard to interpretation. In a college geometry class, **proof** can be thought of as a convincing deductive argument relying on explicit reference to axioms or previously proven theorems. Since a model of an axiom system is an interpretation of the undefined terms that satisfies the axioms, every theorem translates to a true statement in a model. Therefore, if a mathematical statement turns out to be false in a model, then the statement cannot be a theorem (i.e. it cannot be proved from the axioms). Moreover, just as models can be used to show that a statement cannot be proven, they can show that a statement cannot be disproven. That is, demonstrating a model where a statement holds shows that the negation of the statement is not a theorem. Models, therefore, serve as a sort of laboratory for geometric conjectures and can be a powerful tool for exploring the properties of an axiom system.

Students need to understand what the axioms mean, and then they can try to convince someone that a theorem is true whenever the axioms are true [see SLO 1 on Proof]. As in other parts of the geometry curriculum, we see a trade-off between trying to be as rigorous as possible and trying to be developmentally appropriate. Because this is the main place for considering axiom systems in the college math curriculum, understanding of the elements of an axiom system (axioms, models, theorems, interpretations, undefined terms) should be an explicit learning goal. Considering interpretations where axioms do not hold is a good starting point. It encourages students to grapple with what the axioms actually mean, their distinction from other axiomatic elements, and their role of being a foundation of mathematical systems. This is therefore a natural place to bring non-Euclidean geometries into the picture [see SLO 9].

When choosing how to introduce an axiom system, instructors must balance the need to establish expectations for axiomatic proof with the need to understand significant, non-obvious geometric results in a reasonable amount of time. In many classes, students start out with a very simple axiom system of basic facts true in almost any geometry and then proceed to prove theorems from them. A common approach is to introduce an axiom system consisting of Euclid's axioms without the parallel postulate, which is used to develop a "neutral geometry.".This has the advantage of making the axioms simple enough to focus on the logic of building deductive inferences using them. One strategy for motivating students to reason axiomatically about neutral geometry is to introduce problems with rules in which undefined terms "points" and "lines" have been replaced either with nonsense terms (“Every Fo has two Fes.”) or with some other context (“Every club has at least two members.”) as is done in Greenberg (2008). This encourages students to reason from the stated rules rather than using their geometric intuition.

When considering an axiomatic development of geometry, it is important to consider the developmental readiness of one’s students for reasoning abstractly. Although GeT students should have learned some elements of the Euclidean geometric system in their K-12 geometry curriculum, it can be helpful if the instructor starts with lower Van Hiele level tasks to scaffold students’ development of geometric reasoning and proof, especially with the consideration that GeT students are often from different STEM majors with varied prior knowledge. Varying expectations of rigor and abstraction can also provide opportunities for assessing students' development of deductive reasoning (*e.g.*, their application of logic) that is not purely axiomatic. Highlighting differences in expectations for justifications can help to solidify students’ understanding of the elements of axiom systems and proofs.

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SLO 5: Understand the role of definitions in mathematical discourse.

Summary
In a geometry class, definitions can be a fruitful area for students to explore. Students can propose their own definitions for geometric objects and relationships, engage in class discussions about mathematical definitions versus vague descriptions, and compare and contrast definitions that refer to different properties. Determining whether and when definitions have equivalent meanings prepares prospective teachers for the varieties of geometric definitions they may encounter in teaching secondary geometry. Prospective geometry teachers should understand the role of precision in definitions for geometric terms and relationships, including understanding that some geometric terms and relationships must remain undefined. They should also understand that there are a variety of acceptable choices for some geometric definitions and that these choices can influence the structure of proofs. For example, proving that two lines are parallel because they do not intersect can be very different from proving that they are parallel because they are everywhere equidistant.

Narrative
The role of definitions in mathematics is a rich area for discussion in a GeT course. In many math classes, definitions are given by the textbook or teacher. However, in a geometry class, students can propose their own definitions for elementary concepts, such as a square, a triangle, a circle, or even a straight line. They can engage in class discussions about verifiable mathematical definitions vs. vague descriptive definitions, and they can compare and contrast definitions with different properties included. For example, when asked to define what a rectangle is, one student might say it is a quadrilateral with four equal angles; another might say it is a quadrilateral that has at least three right angles and does not have four equal sides; another might say it is a quadrilateral with reflection symmetry across the perpendicular bisectors of its sides; yet another might say it is a quadrilateral with four congruent angles and two pairs of congruent parallel sides.

Classes can have rich discussions regarding both the equivalence and the quality of proposed definitions. Criteria for the quality of definitions could include (1) the use of commonly understood words or previously defined terms, (2) accurately describing what is being defined, and (3) including no superfluous information. One strategy to convey the need for the first criterion is to “define” two “nonsense” words with definitions that refer to one another and, thus, have no meaning. Some definitions must involve undefined terms to avoid infinite regress. A strategy to convey this is to ask students to come up with a definition of a familiar object and prompt them to define the terms they use in their definition.

Choices for definitions necessarily set the context for engaging in activities around proof. For example, proving that two lines are parallel because they do not intersect can be very different from proving that they are parallel because they are everywhere equidistant. GeT students can also consider how changes in the assumptions within a geometric definition can lead to changes in the interpretations of other terms involving that definition. For example, a circle is often defined as the set of all points in a plane that are equidistant to a given point. If a geometry adopted the Euclidean metric for distance, then the property that distinct circles have a finite number of intersections holds; however, this property is not maintained with the Taxicab metric (Krause, 1975).

The logical consequences of statements involving a definition include the assumed meanings for terms within a definition, as well as the axioms of the system [see SLO 4]. For those who include significant non-Euclidean topics [see SLO 9]. there is an opportunity to investigate the same definitions using different models. For example, there are no quadrilaterals with four right angles on the surface of the sphere or on the hyperbolic plane, but there are still quadrilaterals with reflective symmetry over the perpendicular bisectors of their sides. On the hyperbolic plane, there are lines that do not intersect but are not everywhere equidistant. On the sphere, there are not any lines that do not intersect, but there are still lines that make equal corresponding angles with a transversal. Some instructors have been surprised to discover that when students have spent time in class exploring definitions and they are then given a new space to explore on their own, they can productively spend weeks exploring the implications of potential definitions. For instance, what is a circle on the surface of a cone? If a circle is defined as the set of all points obtained by going a fixed distance from a given center in all directions, we get different circles than if a circle is defined as a figure with constant curvature, which is, in turn, different than if a circle is defined as a closed figure such that every straight line segment from the center to the boundary is the same length. Each of these types of circles has different properties that students can explore.

The taxonomy of geometric objects is closely tied to definitions, and the exercise of classifying objects also helps GeT students attend to the ramifications of adopting different definitions and become prepared to support prospective students' reasoning at different Van Hiele levels (see, e.g., Burger & Shaughnessy, 1986). In elementary school, students are taught how to identify and classify different quadrilaterals as rectangles, rhombi, squares, or none of the above. As definitions become formalized in middle and high school geometry, they become associated with increasingly generic representations. It is in the GeT course that students consider the results of adopting alternative definitions for geometric terms. For example, a trapezoid is typically defined *inclusively* in college geometry courses (a trapezoid is a quadrilateral with *at least* one pair of parallel sides). Still, it is sometimes defined *exclusively* in elementary and secondary courses (a quadrilateral with *exactly* one pair of parallel sides). Other terms commonly encountered in secondary geometry for which GeT students could discuss the consequences of adopting definitions supporting exclusive or inclusive meanings include: whether coincident lines are types of parallel lines, whether kites are types of rhombi, and whether the identity transformation is considered to be a type of rotation or a type of translation. Determining whether and when definitions have equivalent meanings and the consequences of adopting exclusive or inclusive definitions prepares GeT students for the variety of geometric definitions they may encounter in teaching secondary geometry.

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SLO 6: Effectively use technologies to explore geometry and develop understanding of geometric relationships.

Summary

Geometry courses need to utilize technology to help develop geometric reasoning and deepen students' understanding of geometric concepts and relationships. Two types of technologies that are beneficial for teaching geometry are dynamic geometry environments (DGEs) and digital proof tools (DPTs). DGEs support student understanding by allowing students to explore properties of geometric figures dynamically, which provides advantages over using paper and pencil. These explorations help students generate their own conjectures, test their conjectures, and provide justification and understanding for theorems. DGEs have been an important tool for teaching geometry since the 1990s. DPTs are an emerging technology that provide students with interactive figures to manipulate and opportunities to practice writing proofs with immediate feedback.

Narrative

Technology can help develop students' understanding of geometry across grade levels and through a variety of aspects. Therefore, the GeT course should utilize technology so students can explore, conjecture, and develop an understanding of geometric relationships. Dynamic geometry software has been shown to help students develop geometric reasoning (Hadas, Hershkowitz, & Schwarz, 2000; Hoyles & Jones, 1998; Jones, 2000).

In addition to using technology to learn geometry, GeT students who are preservice teachers need to understand best practices in using technology in a geometry classroom. Based on a systematic analysis of a set of frameworks related to learning or teaching mathematics with technology, McCulloch et al. (2021) identified four categories of frameworks (Figure 1) that can "inform the framing of how teachers learn to use technology for instruction" (p. 331). Preservice teachers need varied experiences with all four categories, but it is essential that GeT courses have students utilize technology for the learning of mathematics (i.e., first branch of Figure 1) which is the foundation of all the categories. Depending on the audience in the GeT course, GeT instructors may include the other three categories somewhere in their preparation program to prepare PSTs to be able to utilize technology for the teaching of mathematics.

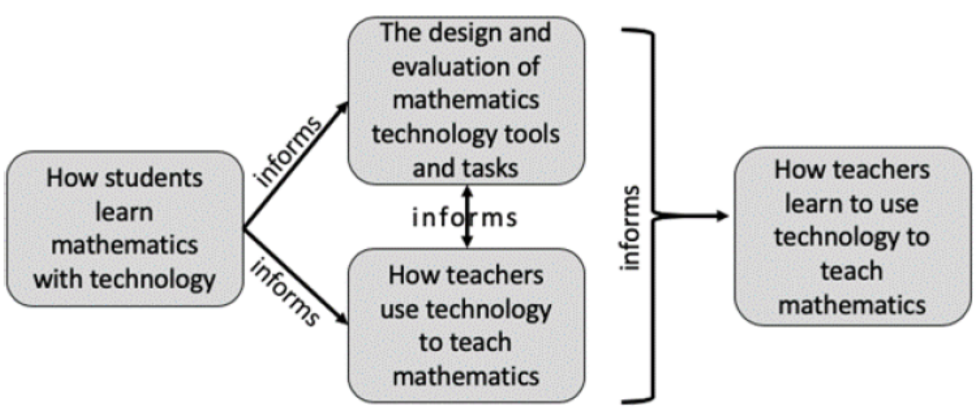

Figure 1: Four Framing Categories Related to Preparing Teachers

to Teach Mathematics with Technology (McCulloch, et al, 2021, p. 331)

All GeT students need experience as learners with technology. Preservice teachers also need to experience reflection regarding the design of tasks that utilize technology. GeT courses are an ideal place for preservice teachers to experience authentic and meaningful use of technology. The main category of technology that has been widely used by both the GeT and secondary instructors is dynamic geometry environments (DGEs). Emerging technology such as digital proof tools (DPTs) is also introduced because of its potential to be beneficial for teaching geometry as its prominence continues to evolve.

Dynamic Geometry Environments

Dynamic geometry environments (DGEs) refer to geometry software that supports the "continuous real-time transformation often called 'dragging'" (Goldenberg & Cuoco, 1998, p. 351). The dragging feature allows the user to change certain elements (e.g., a point) in a constructed geometric figure and observe the change of the corresponding geometric relationships in the figure. The constructed figures are referred to as "draggable" or "moving" figures, which can provide the user with opportunities to experience "motion dependency" and further explore "logical consequence between properties within the geometrical context" (Mariotti, 2014, p. 159). Geometer's Sketchpad, GeoGebra, and Desmos are the most commonly used DGEs (Table 1).

The development of DGEs in the 1990s opened up significant new possibilities for teaching and learning geometry. They offer ways to experiment with lots of possible figures at once rather than just one static figure, to make exact measurements easily, and to incorporate geometric exploration and making conjectures. For example, DGEs can be used to explore properties of various types of quadrilaterals so students understand how quadrilaterals are related to each other. DGEs also provide an easy way to demonstrate transformational geometry, with built-in tools for translating, rotating, reflecting, and dilating objects. In order for students to solidify their understanding of geometry, DGEs should be included in a GeT course.

Emerging Technology - Digital Proof Tools

Proof and reasoning lie at the heart of any geometry course, and GeT courses are no exception. Many students struggle with proof and recently developed digital tools have been created to help students with the proof process. As an emerging category of technology, DPTs allow the instructors to create and edit geometry proof problems and provide students with feedback or hints to facilitate their proof-writing process. Two-column proofs are usually supported by DPTs because of their neat organization and clear logic flow. Drawing features can help students identify the given conditions or add auxiliary lines in a diagram. CanFigureIt Geometry®, FullProof, and Proof Companion are some of these DPTs (Table 2).

An advantage of DPTs is that they provide opportunities for students to practice two-column proofs and get instant feedback. It is becoming a common practice to use online homework in mathematics courses at the secondary level, and students like it because of the instant feedback. Geometry proof problems are not included in typical online homework systems so DPTs provide a solution to this issue.

Table 1: ***Some Commonly Used Dynamic Geometry Environments***

	Software	Description of Tool
	Desmos	Basic geometry construction and transformation tools.
DGEs	GeoGebra	More powerful geometry construction and transformation tools (compared to Desmos). Visualizing 3D objects and creating 2D nets that correspond to 3D objects. It includes activities with non-Euclidean geometries.
	Geometer's Sketchpad	The first well-known DGE that was created by Key Curriculum Press. It has comparable capabilities as GeoGebra but requires purchasing a license to access all features. The last commercial version of Sketchpad has been discontinued, but a new free version is in development.
	Spherical Easel	This software is currently being updated and the link will be added later. Allows the users to construct shapes, rotate, and make measurements on a sphere.
	WebSketchpad	Similar to Geometer's Sketchpad, but allows teachers to select which tools from the Tool Library can be used for a specific activity.

Table 2: ***Some Emerging Digital Proof Tools***

	Software	Description of Tool
	CanFigureIt Geometry®	Teachers can select and assign geometry proof tasks through a virtual class platform. Gives students continuous feedback and guidance during the proving process.
DPTs	Full Proof	Personalized proof-writing tool using interactive diagrams and an equation editor. Provides detailed feedback, hints, and fine-grained evaluation.
	Proof Companion	Teachers can create or edit a geometry proof to share with students and track their progress. Students simply drag and drop the statements and reasons to their proper position to have their work instantly graded.

As technology continues to improve, there will likely be new tools to help with teaching and understanding geometry so this list is not exhaustive. Thus, GeT courses will need to evolve so that we can continue to prepare effective secondary geometry teachers.

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<u>SLO 7: Demonstrate knowledge of Euclidean geometry, including the history and basics of Euclid's *Elements* and its influence on math as a discipline.</u>

Summary

The study of many mathematical subjects can be illuminated by looking at their histories; this is especially true of geometry. Euclidean geometry is named after Euclid, the Greek mathematician who lived in Alexandria around 300 BCE. Euclid systematized the knowledge of geometry and included it in thirteen books called *The Elements*. In *The Elements*, Euclid set out a sequence of Definitions, Postulates (axioms for geometry), Common Notions (axioms common to all mathematical subjects), and Propositions (theorems derived logically from the preceding materials). For most of the 2400 years since it was written, it was considered to be an essential text and the gold standard of mathematical rigor. Students need to know this history to place modern ideas about proof into context and to understand mathematics as a human endeavor.

Narrative

Euclidean geometry is named after Euclid, the Greek mathematician who lived in Alexandria around 300 BCE. Euclid synthesized what was known at the time about Euclidean geometry into the thirteen books of *The Elements*. In *The Elements*, Euclid sets out a sequence of definitions, postulates (axioms for geometry), common notions (axioms common to all mathematical subjects), and propositions (theorems derived logically from the preceding materials). It is the "oldest extant large-scale deductive treatment of mathematics" ("Euclid's *Elements,"* 2022). For most of the 2400 years since it was written, it was considered to be an essential text that any educated person would have studied; it is only in the last 150 years that this was no longer true. Likewise, for most of that time, it was considered to be the gold standard of mathematical rigor; again, it is only in the last 150 years that its rigor has been surpassed.

Many mathematical disciplines can be illuminated by considering their histories, and this is especially true of geometry. Students need to know this history to place modern ideas about proof into context. A geometry class is likely where students will first encounter the notion of axiomatic proof, and for some of them it may be their only encounter with it. [See SLO 4 on Axiomatic Systems.] Euclid's first three postulates also set the stage for constructions using straightedge and compass [see SLO 8 on Constructions].

One particularly interesting piece of this history concerns Euclid's fifth Postulate: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles" (Euclid, 1956). Over the years, this postulate was considered inferior to the others, and many mathematicians tried to prove it from the others. It was only in the early nineteenth century that it was discovered that there is a geometry in which the other postulates are true, but this one is false. This is how hyperbolic geometry was discovered [see SLO 9 on non-Euclidean geometry].

Most students will not have seen Euclid's fifth postulate in its original form but would likely know something more like Playfair's postulate that given a line and a point not on the line that there is a unique line through the point parallel to the given line. Proving the logical equivalence to Euclid's fifth postulate is a good homework problem.

One question that remains relevant to modern geometry teachers is to what extent a geometry course should cover Euclid's actual writings and methods, or to what extent they should be replaced by simplified treatments (as is done in many high school geometry books) or more technical rigorous methods (such as those based on Hilbert's *Foundations of Geometry*). As discussed above, for most of the time since it was written, almost all educated people studied *The Elements*. The debate about what treatments of geometry might be better than Euclid for modern students started in the mid-nineteenth century. There was enough debate that in 1879, Charles Dodgson, the mathematician better known as Lewis Carroll, published a book entitled *Euclid and his Modern Rivals*, which argued that Euclid's treatment of geometry was superior for teaching students than any other treatment then proposed. This debate continues to this day.

One reason to consider teaching Euclid's original proofs, besides their historical interest, is that they are generally at the right level of sophistication for students when more modern treatments that are considered more rigorous might also be too complicated for students. One theory of cognitive development espoused by Jean Piaget, among others, states that "ontogeny recapitulates phylogeny," that is, individual development often follows a similar path to the historical development of a subject (Gould, 1977). Given the outsized role that Euclid's work has played in the historical development of geometry, it is not surprising that it might be at just the right level of sophistication for many students.

Looking at Euclid also makes it natural to consider where Euclid's treatment may have gaps. For example, Euclid's first postulate says that it is possible "to draw a straight line from any point to any point," but if you look at the way he uses it, he really assumes that the line drawn is unique. Looking at his proofs through the lens of a geometry where any two points can be connected but not necessarily uniquely, such as on the sphere, makes this issue immediately clear.

Euclid also introduces the idea of superposition to prove the Side-Angle-Side congruence condition for triangles. This argument was criticized because of the assumptions his proof makes about transformations without stating them explicitly. This can lead to a discussion of the transformations approach to Euclidean geometry.

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SLO 8: Be able to carry out basic Euclidean constructions and justify their correctness.

Summary

Traditional geometric constructions are those done exclusively with a compass and straightedge. However, the term can be used more generally to include other tools and manipulatives such as paper-folding (e.g., using origami or patty paper), dynamic geometry environments, or transparent mirrors (e.g., MIRAs). Constructions remain an essential part of Euclidean geometry and therefore of a GeT course. Constructions support the development of mathematical thinking in several essential ways: they provide a natural opportunity for making mathematical arguments, encourage the use of precise mathematical language in communication, impart a sense of where assumptions in building mathematical systems come from, open discussions of the historical development of geometry—especially the work of Euclid, and give future teachers experience with the curriculum they will be expected to teach.

Narrative

Geometric constructions support the curriculum in a GeT course and the development of mathematical thinking in several essential ways:

- they provide a natural place for making a mathematical argument;
- they encourage practice in using precise mathematical language when describing a construction;
- they provide the students a sense of where assumptions in building mathematical systems come from;
- they provide openings for discussion of the historical development of geometry, especially to the work of Euclid;
- and they give students experience with the curriculum they will be expected to teach.

For these reasons, constructions remain an essential part of any GeT course curriculum. It should be noted that within the context of geometry, the term "construction" most often refers to traditional straightedge and compass constructions, and this is, generally, the assumption that we make here. However, the term can be used more generally to include other tools and manipulatives such as paper-folding (e.g. using origami or patty paper) or MIRAs™. In this context, using an external tool for the creation of an ideal geometric object whose success is argued rigorously might be reasonably considered a "construction." Indeed, the Common Core Standards for School Mathematics (NGA/CCSSO, 2010) includes the following in its "Congruence Standard":

> *G-CO.12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

A GeT instructor giving even a cursory treatment of constructions should consider including the constructions listed above in CCSSM *G-CO.12* in their course. Having students discover the constructions themselves can be a powerful mathematical experience; having them present the constructions to the class offers opportunities for practice in using mathematical terminology and in making rigorous and logical arguments. The question, “How do we know that this construction is correct?” arises as a matter of course. In this way, constructions naturally reinforce SLO 1 [Proof].

Constructions—especially those using straightedge and compass—support SLO 7 [Euclid’s *Elements*] as well. Not only are many of the “standard” constructions listed above given in Book I of the *Elements*, they also illustrate the discipline of mathematics as put forth by Euclid. The standards Euclid used to build geometry form the basis for the current standards of rigor used in mathematics. In building geometry, Euclid set out to establish results, many of which were known during his time, by starting with as few assumptions as possible. By viewing geometric results as constructions from two simple tools, namely a straightedge and a compass, Euclid was able to create a foundation on which all his propositions rest. Because of his approach, Euclid had to rely heavily on the power of logical reasoning. Moreover, he needed a good grasp of concepts/definitions in order to convince himself and others that his constructions were able to do what they were purported to do. Employing Euclid’s approach to building geometry allows us to view a field of mathematics (geometry, algebra, analysis, etc) as a unified whole and it has provided us with numerous beautiful results, many of which have found practical applications in the world.

Instructors looking for ways to incorporate transformations [see SLO 10] into their GeT course will find that constructions offer a useful pathway, as the fundamental transformations (reflection, rotation, translation) can be represented as straightedge and compass constructions. In addition, for those GeT courses that specifically emphasize transformations, constructions offer opportunities for mathematically rich explorations. For example, an instructor could ask students to construct the composition of two reflections of a triangle across distinct lines (either parallel or intersecting). Identifying the single rigid motion that has the same effect requires careful construction, precision of language, and the use of cases. Nevertheless, the problem is an accessible one, and it can naturally lead to a discussion of the group of isometries.

Finally, for an instructor who intends to include the exploration of models of non-Euclidean geometry [see SLO 9] in their course, work with Euclidean constructions is time well spent. Since many models of hyperbolic and spherical geometry reside in Euclidean geometry, constructions in those models rely on Euclidean counterparts. For example, constructing a “line” through two points in the Poincare half-plane involves finding a perpendicular bisector of a Euclidean line. With the benefit of dynamic geometry software [see SLO 6], even some of the complex constructions in non-Euclidean geometry become accessible, once one understands the fundamental Euclidean constructions.

References

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SLO 9: Compare Euclidean geometry to other geometries such as hyperbolic or spherical geometry.

Summary

Just as visiting another country can offer one a richer perspective on their own culture, the study of non-Euclidean geometries can help students to develop a deeper understanding of Euclidean geometry. The term "non-Euclidean geometry" is interpreted broadly here, referring to any geometry different from Euclidean, including spherical, hyperbolic, incidence, and taxicab geometries, among others. The choice of which non-Euclidean geometries to consider might depend on the demands of a particular GeT course, but all non-Euclidean geometries offer rich opportunities to explore and visualize novel and engaging worlds. Learning the properties of non-Euclidean geometries puts preservice teachers in the position of their students who may be learning Euclidean geometry for the first time. Moreover, non-Euclidean geometries challenge student assumptions about what is "true" or "obvious" in Euclidean geometry (e.g. the shape of a parabola or the angle sum in a triangle). In this way, exploring non-Euclidean geometries naturally encourages questions of "Why?" and "How?" and supports the development of mathematical thinking, especially the need for justification.

Narrative

Just as visiting another country can offer us a richer perspective on our own culture, so too can the study of non-Euclidean geometries help students to develop a deeper understanding of Euclidean geometry. While it is natural for students to be uncomfortable working in a geometry that varies from their intuition, non-Euclidean geometries afford an opportunity to explore and visualize novel worlds that can engage their imagination. In addition, learning the rules of these geometries puts students that plan to teach in the position of their students who may be learning Euclidean geometry for the first time. The choice of which different non-Euclidean geometries to consider might depend on the demands of the GeT course, but all offer new perspectives on familiar geometric objects and relationships.

In our everyday experience, we regularly encounter multiple geometries. Buildings tend to be Euclidean. We expect floors and ceilings to be planar. Outside, the horizon reminds us that we live on a sphere. Our visual field routinely processes distant objects as smaller than comparably sized things that are nearby, just as they could be represented in projective geometry. In the car, we measure distance with a taxicab metric. Fans of science fiction may even encounter images and ideas of hyperbolic geometry. Notably, non-Euclidean geometries can be viewed through two different lenses: geometrically, as spaces that are physically different from Euclidean space, or axiomatically, as spaces in which different axioms are true [see SLO 4].

The amount of time devoted to non-Euclidean geometries can vary widely depending on factors including audience, instructor preference, and institutional expectations. For a class consisting primarily of preservice teachers, a substantial amount of Euclidean content is necessary, though at least some non-Euclidean geometry is recommended. In a comparative geometries course, it would be natural to consider several different non-Euclidean geometries, while a class that focuses primarily on Euclidean geometry might include a brief survey of some non-Euclidean examples or focus on one flavor for a longer period of time. In any case, what follows are some of the learning opportunities offered by each.

Incidence Geometries are useful for getting a sense of how theorems follow from a set of axioms. These involve a reduced set of axioms and perhaps make it easier to introduce some principles of proof-writing [see SLO 1] in that context. Taxicab geometry is an easily described alternate geometry that can lead to rich mathematical exploration. Here we note that when we speak of "non-Euclidean geometry," we mean this broadly, referring to geometries that are different from our usual notion of Euclidean two-or three-dimensional space. In taxicab geometry, we change our usual definition of distance in the plane. Rather than using a Pythagorean measurement, we measure the distance between two points as the sum of the absolute differences of their Cartesian coordinates. This radically changes the form of objects that are defined in terms of distance. For example, a circle (the set of points at a given distance from a given point) no longer appears round. Ellipses, hyperbolas, and parabolas provide an even greater challenge!

Spherical geometry offers the advantage of being (fairly) easy to visualize (or hold in your hand). As a more accurate representation of the surface of the planet than a flat Euclidean world, it has relevance. An introduction to spherical geometry immediately challenges our understanding of the undefined term "line" [see SLO 5 on definitions] and our belief that between any two points there can be drawn a unique line. Other explorations might have students consider parallel lines on the sphere or the angle sum of a triangle.

Taxicab and spherical geometry serve well as examples of non-Euclidean geometries that can be explored at any point in a GeT course. Hyperbolic geometry can be as well, though its close relationship to Euclidean geometry is, perhaps, best appreciated when students are more experienced with axioms and axiomatic systems [see SLO 4]. Hyperbolic geometry differs from Euclidean geometry only in a parallel postulate. In Euclidean geometry, we assume there is exactly one parallel through a given point not on a given line. In hyperbolic geometry, we adopt a different parallel postulate, so that there are multiple lines through a given point parallel to a given line. Changing this axiom is, in fact, how hyperbolic geometry was first developed historically[4] [see SLO 7]. Moreover, we can simply remove that axiom altogether to end up with a third geometry: Neutral geometry. Comparing the mathematical properties of these three geometries and their interplay leads to rich discourse. It is also worth noting that although hyperbolic geometry may arise most naturally from this axiomatic change, it can also be viewed

[4] See (Hofstadter, 1999, Ch. 4, pp. 88–93), for an accessible discussion of this history written for a general audience.

geometrically as a space of constant negative curvature. In this sense, it provides an instructive example of a non-Euclidean geometry having properties different from Euclidean geometry.

The relationship between Euclidean, hyperbolic, and Neutral geometry can be made explicit by proving the equivalence of parallel postulates in Neutral geometry. For example, transitivity of parallelism ("Two distinct lines each parallel to a third line are parallel to each other.") is logically equivalent to Euclid's fifth postulate in Neutral geometry. Proving that equivalence, or one similar, can be a valuable experience by strengthening student understanding of proof [See SLO 1.]. These proofs are demanding but generally relatively brief.

Rectangles (quadrilaterals with four right angles) are among our most familiar geometric objects. However, while the existence of rectangles can be easily established in Euclidean geometry, it cannot be proven that they exist in hyperbolic geometry or in spherical geometry. In both of those cases, a pair of lines can share, at most, one common perpendicular line—making a rectangle impossible. In neutral geometry, we can neither prove nor disprove their existence. Likewise, we can show that similar, non-congruent triangles do not exist in hyperbolic geometry. Examples such as these differentiate between the geometries and demonstrate the necessity of Euclid's fifth postulate. In this way, they can help strengthen student understanding of axiom systems [see SLO 4.].

Hyperbolic geometry offers opportunities for students to strengthen their facility with geometric straightedge and compass constructions [see SLO 8.] through an exploration of hyperbolic geometry models. Since both the Poincaré and Klein models of hyperbolic geometry are situated within Euclidean geometry, constructions of "lines" and "perpendiculars" in these models translate to Euclidean constructions. These constructions can cover a range of complexity, from the trivial (e.g., constructing a "line" in the Klein disk) to the highly involved (e.g., "dropping a perpendicular" in the Poincaré disk). Dynamic geometry software [See SLO 6.] is an excellent resource here, as there is a wealth of online tools available that automate some of the most difficult constructions. This has the added benefit of encouraging students to engage in higher-order thinking on constructions in ways that were not possible only a few years ago.

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SLO 10: Use transformations to explore definitions and theorems about congruence, similarity, and symmetry.

Summary
Two main types of transformations arise in GeT courses: isometries (also known as congruence transformations) and similarity transformations. Isometries include reflections, rotations, translations, and glide reflections; a similarity transformation is the composition of an isometry and a dilation. A GeT course can contain a dedicated unit on transformations, or transformational concepts can be integrated throughout the course (even to the extent of a purely axiomatic treatment). In order to enhance and facilitate prospective teacher learning, familiar function notation can be incorporated in an introduction to transformations. This may lead to a better understanding of a sequence of transformations as a composition. While some GeT courses may begin with an informal approach to the understanding of reflections and rotations and extend the concepts of symmetry to study geometric shapes using transformations, others may instead begin solely with sequences of reflections, which can be used to generate all other isometries of the plane.

Narrative
Typically, there are three different approaches from which instructors can choose in order to embed transformation geometry learning outcomes into a GeT course. Instructors may choose to teach a GeT course using a formal transformation approach; they may choose to include a dedicated transformation unit within a course that has a predominantly traditional Euclidean focus; or, they may integrate transformation approaches and concepts throughout the course offering.

There are two main types of transformations that arise in GeT courses: isometries, also known as congruence transformations (mappings that preserve both angle measure and segment length), and similarity transformations (mappings that preserve angle measures and proportionality of segment lengths). The set of isometries of the plane form an infinite non-Abelian group. Moreover, each element of the group can be classified as the identity transformation, a reflection, rotation, translation, or a glide reflection, and the composition of, at most, three reflections can be used to generate the other isometries. Depending on abstract algebra prerequisites, instructors may choose to highlight and formalize group properties and connections to finite symmetry groups.

It is recommended that GeT instructors build on familiar function notation when introducing transformations. This may lead to a better understanding of sequences of isometries as compositions of isometries. This notation may also make it easier for students to use rigid motions to express symmetry. GeT instructors can take advantage of looking at proofs through multiple approaches (Euclidean, analytic, transformational) to deepen students' understanding of specific theorems. For example, students can be prompted to compare other strategies after proving the base angles of an isosceles triangle are congruent by using the concept of symmetry.

Some GeT instructors may begin with an informal approach to the understanding of reflections and rotations and extend the concepts of symmetry to study geometric shapes using transformations (translations, rotations, and reflections) and combinations of them. The definition of congruence is then conveyed in terms of rigid motions: two figures, A and B, are considered congruent if, and only if, there exists a sequence of rigid motions, r, that superimposes figure A onto figure B, that is r(A)=B. Instructors may consider pointing out advantages of this definition of congruence, most notably that the definition applies to all congruent figures, not just congruent triangles; they may also highlight that transformations act on the entire plane. Similarity is defined analogously in terms of dilations and rigid motions.

The notion of a translation typically will be introduced with directed line segments and a rotation with directed angles, though some instructors might take the opportunity to more deeply explore the concepts in terms of vectors, matrices, and coordinate geometry. Instructors may instead choose to begin solely with sequences of reflections, which can be used to generate all other isometries of the plane. Each line of the plane is associated with a reflection that satisfies two properties: (1) every point on the line is fixed by the transformation and (2) the line is the perpendicular bisector of the segment connecting any point not on the line and the point's image under the transformation. By exploring the images of points and figures resulting from sequences of reflections about parallel and intersecting lines, GeT students can discover and establish relationships with translations, rotations, and glide reflections. Instructors may choose to also have students explore, informally, orientation preserving/reversing properties and the aspects of the group structure (associativity of composition, existence of identity and inverses, and non-commutativity).

GeT instructors have reported that students sometimes struggle with understanding fixed point properties of transformations. A case in example is when a segment is rotated a specified number of degrees about a center of rotation when the center of rotation (the fixed point) is not on the segment. Even though the center of rotation is specified, students often choose one end-point of the segment as a center of rotation and use it as the fixed point. Using activities and technology that allow students to experience multiple examples of such properties can be included in a GeT course [see SLO 6 on use of dynamic geometry software.] to clarify such misunderstandings.

It is important that GeT students move beyond recognizing the properties of transformations and learn to reason with them. The Common Core State Standards for Mathematics (2010) require using transformations to justify the triangle congruence and similarity criteria (e.g., SAS, AA). Although the Common Core does not specify transformational proofs beyond this, Douglas and Picciotto (2018) provide a guide and activities for transformational proof in high school geometry for teachers and curriculum developers; the guide recommends a high school geometry course using both traditional Euclidean and transformational approaches, beyond congruence and similarity criteria for triangles. St. Goar and Lai (2021) note ways to move beyond triangle congruence criteria proofs and to incorporate the bidirectionality of the definition of congruence (or similarity). They state that in order to prove figure A is congruent to figure B based on a transformation approach, a student must use two key elements: (1) they must specify a sequence of rigid motions, *r*, from figure A to figure B (NCTM, 2018) and (2) they must justify deductively that the image of figure A under the rigid motions actually is figure B, that is *r*(A)=B. Similarity proofs are constructed analogously.

Instructors may also choose to compare definitions based on properties of transformations with other definitions [see SLO 5]. For example, one could define a kite as a convex quadrilateral for which a diagonal is a line of symmetry and prove properties about its congruent sides and congruent angles that are used as definitions in a traditional approach. The transformational context could be an opportunity to compare different axiomatic contexts [see SLO 4]. For example, without basic assumptions about transformations, traditionally one of the triangle congruence criteria—ASA, SAS, or SSS—must be assumed as an axiom without proof. By contrast, in a transformation context, ASA, SAS, and SSS may all be proved deductively without such assumptions (Venema, 2006). Other opportunities to learn transformational geometry include explorations of tessellations of the Euclidean plane as well as explorations of transformations in non-Euclidean geometries [see SLO 9].

It is important that prospective teachers are exposed to multiple approaches so that they not only are prepared with a solid background in geometry content but also so that they are able to make informed decisions regarding the choices they make in their own classrooms. Additional resources that may assist instructors with this effort include Venema (2006), which has a robust mathematical discussion of transformations, and Henderson & Taimina (2005), which offers an inquiry-based learning approach to transformations and a way to include rigid motions in other geometric contexts such as spherical and hyperbolic geometry.

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